On characters and superdimensions of some infinite-dimensional irreducible representations of $\mathfrak{osp}(m|n)$

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Chiral spinors and self dual tensors of the Lie superalgebra $\mathfrak{osp}(m|n)$ are infinite dimensional representations belonging to the class of representations with Dynkin labels $[0,\ldots,0,p]$. We show that the superdimension of $[0,\ldots,0,p]$ coincides with the dimension of a $\mathfrak{so}(m-n)$ representation. When the superdimension is finite, these representations could play a role in supergravity models. Our technique is based on expansions of characters in terms of supersymmetric Schur functions. In the process of studying these representations, we obtain new character expansions.

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I. INTRODUCTION

Models of supergravity theory [1, 2] are often implicitly or explicitly based upon tensor representations of the orthosymplectic Lie superalgebra $\mathfrak{osp}(m|n)$ [3, 4]. Chiral spinors and self dual tensors of $\mathfrak{osp}(m|n)$ play an important role in such models. These tensors are, however, infinite-dimensional. Nonetheless, the so-called superdimension of these tensors corresponds to the dimension of a finite-dimensional tensor of $\mathfrak{so}(m-n)$ [5] (to be interpreted appropriately when $m-n$ is negative [6]), thus paving the way for new covariant quantization schemes.

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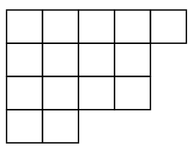
In [5] we initiated the study of this correspondence between certain infinite-dimensional representations of $\mathfrak{osp}(m|n)$ and finite-dimensional representations of $\mathfrak{so}(m-n)$. Let us be more precise. In terms of (the distinguished) Dynkin diagrams of $\mathfrak{osp}(m|n)$, the spinor representation has Dynkin labels $[0,0,\ldots,0,1]$ and the self dual tensor $[0,0,\ldots,0,2]$. In [5], we treated the irreducible representations (irreps) with Dynkin labels $[0,0,\ldots,0,p]$, where p is a positive integer (a convention followed throughout this paper).

In the present paper, we shall first review some of the results of [5], and for this we need to recall some definitions and notations. For all these developments, characters of a class of representations of $\mathfrak{osp}(m|n)$ play a prominent role. Since the Lie superalgebras $\mathfrak{osp}(2m+1|2n)$ and $\mathfrak{osp}(2m|2n)$ both contain the general linear Lie superalgebra $\mathfrak{gl}(m|n)$ as a subalgebra, it is convenient to express the characters of the infinite-dimensional $\mathfrak{osp}$-irreps as an infinite sum of $\mathfrak{gl}(m|n)$ characters (given by supersymmetric Schur functions). In [5] this was done for the irreps $[0,0,\ldots,0,p]$ of $\mathfrak{osp}(2m(+1)|2n)$ (leading to a new character formula for the case of $\mathfrak{osp}(2m|2n)$). In the current paper, we can extend this and obtain new character formulas for irreps of type $[0,\ldots,0,r,p-r]$ for $\mathfrak{osp}(2m|2n)$.

II. DEFINITIONS AND NOTATIONS

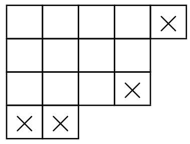
A. Partitions and (super)symmetric functions

We need some basic notions on partitions and symmetric functions, see [7] as a standard reference. A partition $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_n)$ of weight $|\lambda|$ and length $\ell(\lambda)\leq n$ is a sequence of non-negative integers satisfying the condition $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n$, such that their sum is $|\lambda|$, and $\lambda_i>0$ if and only if $i\leq\ell(\lambda)$. To each such partition there corresponds a Young diagram F^λ consisting of $|\lambda|$ boxes arranged in $\ell(\lambda)$ left-adjusted rows of lengths λ_i for $i=1,2,\ldots,\ell(\lambda)$. For example, the Young diagram of $\lambda=(5,4,4,2)$ is given by



The conjugate partition λ' corresponds to the Young diagram of λ reflected about the main diagonal. For the above example, $\lambda'=(4,4,3,3,1)$.

If λ,μ are partitions, one writes $\lambda\supset\mu$ if the diagram of λ contains that of μ. The difference $\lambda - \mu$ is called a skew diagram [7]. For example, if $\mu = (4, 4, 3)$, then the boxes of the skew diagram $\lambda - \mu$ are crossed in the following picture:



A skew diagram is a *horizontal strip* if it has at most one box in each column. The number of boxes of the horizontal strip is its length. The above example is a horizontal strip of length 4.

Partitions are used to label symmetric functions. The Schur functions [7] or S-functions $s_\lambda(x)$ form a $\mathbb{Z}$-basis of the ring Λ_n of symmetric polynomials with integer coefficients in the n independent variables $x = (x_1, x_2, \ldots, x_n)$, where λ runs over the set of all partitions of length at most n. For a partition λ with $\ell(\lambda) \le n$, one has $s_\lambda(x) = \det(x_i^{\lambda_j+n-j})_{1\le i,j\le n}/\det(x_i^{n-j})_{1\le i,j\le n}$. If $\ell(\lambda) > n$, one puts $s_\lambda(x) = 0$.

In terms of two sets of variables $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$, one can define the ring $\Lambda_{m,n}$ of supersymmetric polynomials with integer coefficients [8]. This ring consists of all double symmetric polynomials in x and y (elements of $\Lambda_m \otimes \Lambda_n$) that satisfy the so-called cancellation property (i.e. when the substitution $x_1 = t$, $y_1 = -t$ is made in an element p of $\Lambda_m \otimes \Lambda_n$, the resulting polynomial is independent of t). For a partition λ, one can define supersymmetric Schur polynomials $s_\lambda(x|y)$ belonging to $\Lambda_{m,n}$ [8, 9]. These polynomials $s_\lambda(x|y)$ are zero when $\lambda_{m+1} > n$. Denote by $\mathcal{H}_{m,n}$ the set of all partitions with $\lambda_{m+1} \le n$, i.e. the partitions (with their Young diagram) inside the (m, n)-hook. The set of $s_\lambda(x|y)$ with $\lambda \in \mathcal{H}_{m,n}$ forms a $\mathbb{Z}$-basis of $\Lambda_{m,n}$.

B. Dimension, superdimension, t-dimension

A finite-dimensional irreducible representation of the Lie algebra $\mathfrak{gl}(n)$ is characterized by a partition λ with $\ell(\lambda) \le n$. In terms of the standard basis $\epsilon_1, \ldots, \epsilon_n$ of the weight space of $\mathfrak{gl}(n)$, the highest weight of this representation is $\sum_{i=1}^n \lambda_i \epsilon_i$, and the representation space will be denoted by $V_{\mathfrak{gl}(n)}^\lambda$. Weyl's character formula for such representations yields $\operatorname{char} V_{\mathfrak{gl}(n)}^\lambda = s_\lambda(x)$, where $x_i = \mathrm{e}^{\epsilon_i}$.

Just as the functions $s_\lambda(x)$ are characters of irreducible representations (or simple modules) of the Lie algebra $\mathfrak{gl}(n)$, the supersymmetric Schur functions are characters of a class of simple modules of the Lie superalgebra $\mathfrak{gl}(m|n)$, namely of the covariant representations [8]. For a partition $\lambda \in \mathcal{H}_{m,n}$, the corresponding covariant representation will be denoted by $V^\lambda_{\mathfrak{gl}(m|n)}$. In terms of the standard basis $\epsilon_1,\ldots,\epsilon_m,\ \delta_1,\ldots,\delta_n$ of the weight space of $\mathfrak{gl}(m|n)$, the highest weight of this representation is $\sum_{i=1}^m \lambda_i\epsilon_i + \sum_{j=1}^n \max(\lambda'_j - m, 0)\delta_j$. The main result of [8] is

$$\operatorname{char} V^\lambda_{\mathfrak{gl}(m|n)} = s_\lambda(x|y), \tag{1}$$

where $x_i = \mathrm{e}^{\epsilon_i}$ and $y_j = \mathrm{e}^{\delta_j}$.

Any Lie superalgebra $\mathfrak{g}$ is $\mathbb{Z}_2$-graded: $\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$. A Lie superalgebra module or representation V is also $\mathbb{Z}_2$-graded: $V = V_{\bar 0} \oplus V_{\bar 1}$. In our convention, the highest weight vector v of V will always be an even vector ($v \in V_{\bar 0}$). When V is finite-dimensional, one can speak of the dimension and superdimension of V:

$$\dim V = \dim V_{\bar 0} + \dim V_{\bar 1}, \qquad \operatorname{sdim} V = \dim V_{\bar 0} - \dim V_{\bar 1}.$$

Superdimension formulas for covariant representations of $\mathfrak{gl}(m|n)$ are known [9]. The result depends on whether m is greater than, equal to, or less than n. It can be summarized as follows:

$$\operatorname{sdim} V^\lambda_{\mathfrak{gl}(n+k|n)} = \dim V^\lambda_{\mathfrak{gl}(k)}, \qquad \operatorname{sdim} V^\lambda_{\mathfrak{gl}(m|m+k)} = (-1)^{|\lambda|} \dim V^{\lambda'}_{\mathfrak{gl}(k)}. \tag{2}$$

In particular, when $m = n$, $\operatorname{sdim} V^\lambda_{\mathfrak{gl}(n|n)} = 0$ unless λ is the zero partition (0) (then $V^{(0)}_{\mathfrak{gl}(n|n)}$ is the trivial module with $\operatorname{sdim} V^{(0)}_{\mathfrak{gl}(n|n)} = 1$). Note that (2) implies: when $\ell(\lambda) > k$ then $\operatorname{sdim} V^\lambda_{\mathfrak{gl}(n+k|n)} = 0$; when $\lambda_1 > k$ then $\operatorname{sdim} V^\lambda_{\mathfrak{gl}(m|m+k)} = 0$.

Finally, let us introduce the notion of t-dimension of a Lie (super)algebra highest weight representation V. This is nothing else but a specialization of the character of V, just like the q-dimension [10, Chapter 10]. Recall that the q-dimension of V, with highest weight Λ, is the specialization $F_1(\mathrm{e}^{-\Lambda} \operatorname{char} V)$, where F_1 is determined by

$$F_1(\mathrm{e}^{-\alpha_i}) = q,$$

and the α_i's are the simple roots of the Lie (super)algebra. So this corresponds to a gradation with respect to the simple roots.

The t-dimension is again a specialization $F(\mathrm{e}^{-\Lambda} \operatorname{char} V)$ of the character, but now F is determined in a different way. For a Lie algebra, of which the simple roots are commonly expressed in terms of the standard basis $\epsilon_1,\ldots,\epsilon_n$, one puts $F(\mathrm{e}^{-\epsilon_i}) = t$. For a Lie superalgebra, of which the simple roots are commonly expressed in terms of the standard basis $\epsilon_1, \ldots, \epsilon_m$, $\delta_1, \ldots, \delta_n$, one puts $F(\mathrm{e}^{-\epsilon_i}) = t$ and $F(\mathrm{e}^{-\delta_i}) = t$ for the t-dimension, and $F(\mathrm{e}^{-\epsilon_i}) = t$ and $F(\mathrm{e}^{-\delta_i}) = -t$ for the t-superdimension.

Let us clarify the meaning by means of an example. Consider the orthogonal Lie algebra $\mathfrak{so}(2n+1)$, with simple roots $\epsilon_1 - \epsilon_2, \ldots, \epsilon_{n-1} - \epsilon_n, \epsilon_n$, and the representation V with Dynkin labels $[0, \ldots, 0, p]$, for which the highest weight is $(\frac{p}{2}, \ldots, \frac{p}{2})$ in the ϵ-basis. For this representation, the character reads [11, 12]

$$\mathrm{char}[0, \ldots, 0, p]_{\mathfrak{so}(2n+1)} = (x_1 \cdots x_n)^{-p/2} \sum_{\lambda_1 \leq p,\ \ell(\lambda) \leq n} s_\lambda(x). \tag{3}$$

So the sum is over all partitions λ such that the Young diagram of λ fits inside the $n \times p$ rectangle, of width p and height n. Specializing this character according to F, one finds:

$$\dim_t[0, \ldots, 0, p]_{\mathfrak{so}(2n+1)} = \sum_{\lambda_1 \leq p,\ \ell(\lambda) \leq n} \dim V^\lambda_{\mathfrak{gl}(n)} t^{|\lambda|}. \tag{4}$$

When the character is expressed in terms of Schur functions, as in (3), it yields in fact the branching of the representation according to $\mathfrak{so}(2n+1) \to \mathfrak{gl}(n)$. When the character is specialized as in (4), it is a polynomial in t (or, in case of an infinite-dimensional representation, a formal power series in t) such that the coefficient of t^k counts the dimension "at level k" according to the $\mathbb{Z}$-gradation induced by the $\mathfrak{gl}(n)$ subalgebra of $\mathfrak{so}(2n+1)$.

C. t-dimension for $\mathfrak{osp}(1|2n)$

In this subsection we shall consider the t-dimension for a class of representations of $\mathfrak{g} = \mathfrak{osp}(1|2n)$. Let us first fix some notation [13–15]. In the common basis δ_j for the weight space of $\mathfrak{osp}(1|2n)$, the odd roots are given by $\pm\delta_j$ $(j = 1, \ldots, n)$, and the even roots are $\delta_i - \delta_j$ $(i \neq j)$ and $\pm(\delta_i + \delta_j)$. The simple roots are

$$\delta_1 - \delta_2,\ \delta_2 - \delta_3, \ldots, \delta_{n-1} - \delta_n,\ \delta_n. \tag{5}$$

The character specialization of the previous subsection corresponds to the following $\mathbb{Z}$-gradation of $\mathfrak{g} = \mathfrak{osp}(1|2n)$: $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1} \oplus \mathfrak{g}_{+2}$, where each $\mathfrak{g}_j$ is spanned by the root vectors corresponding to the following roots:

$\mathfrak{g}_{-2}$	$\mathfrak{g}_{-1}$	$\mathfrak{g}_0$	$\mathfrak{g}_{+1}$	$\mathfrak{g}_{+2}$
$-\delta_i - \delta_j$	$-\delta_i$	$\delta_i - \delta_j$	δ_i	$\delta_i + \delta_j$

Note that $\mathfrak{g}_0 = \mathfrak{gl}(n)$.

We will consider infinite-dimensional highest weight representations V of $\mathfrak{g}$, such that the action of $\mathfrak{g}_0 = \mathfrak{gl}(n)$ on the highest weight vector v of V corresponds to a finite-dimensional $\mathfrak{g}_0$ module V_0. Then the $\mathbb{Z}$-gradation of $\mathfrak{g}$ induces a $\mathbb{Z}$-gradation of V:

$$V = V_0 \oplus V_{-1} \oplus V_{-2} \oplus \cdots$$

in terms of finite-dimensional $\mathfrak{g}_0$ modules, and the t-(super)dimension gives

$$\dim_t(V) = \sum_{i=0}^{\infty} \dim V_{-i}\ t^i, \quad \mathrm{sdim}_t(V) = \sum_{i=0}^{\infty} \dim V_{-i}\ (-t)^i = \dim_{-t}(V). \tag{6}$$

For reasons that will become clear, we will consider the irreducible highest weight representation with highest weight given by $(-\frac{p}{2}, -\frac{p}{2}, \ldots, -\frac{p}{2})$ in the δ-basis. For this representation, the Dynkin labels are $[0, 0, \ldots, 0, -p]$. The structure and character of this representation have been determined in [16]. Using the notation $x_i = \mathrm{e}^{-\delta_i}$, one has:

$$\mathrm{char}[0, 0, \ldots, 0, -p]_{\mathfrak{osp}(1|2n)} = (x_1 \cdots x_n)^{p/2} \sum_{\lambda,\ \ell(\lambda)\leq p} s_\lambda(x). \tag{7}$$

This is an infinite sum over all partitions of length at most p. Since $s_\lambda(x) = 0$ if $\ell(\lambda) > n$, the sum is actually over all partitions satisfying $\ell(\lambda) \leq \min(n, p)$. Thus:

$$\dim_t[0, 0, \ldots, 0, -p]_{\mathfrak{osp}(1|2n)} = \sum_{\lambda,\ \ell(\lambda)\leq \min(n,p)} \dim V^\lambda_{\mathfrak{gl}(n)} t^{|\lambda|}. \tag{8}$$

This infinite sum can be rewritten in an alternative form, see [5].

III. SUPERDIMENSIONS FOR $\mathfrak{osp}(2m+1|2n)$

Consider the Lie superalgebra $B(m, n) = \mathfrak{osp}(2m + 1|2n)$, with the distinguished set of simple roots in the ϵ-δ-basis [13, 15]

$$\delta_1 - \delta_2,\ \ldots, \delta_{n-1} - \delta_n,\ \delta_n - \epsilon_1,\ \epsilon_1 - \epsilon_2,\ \ldots, \epsilon_{m-1} - \epsilon_m,\ \epsilon_m. \tag{9}$$

Also in this case there exists a useful $\mathbb{Z}$-gradation of $\mathfrak{g} = \mathfrak{osp}(2m + 1|2n)$: $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1} \oplus \mathfrak{g}_{+2}$, where each $\mathfrak{g}_j$ is spanned by the root vectors corresponding to the following roots:

$\mathfrak{g}_{-2}$	$\mathfrak{g}_{-1}$	$\mathfrak{g}_0$	$\mathfrak{g}_{+1}$	$\mathfrak{g}_{+2}$
$-\delta_i - \delta_j$	$-\delta_i$	$\delta_i - \delta_j$	δ_i	$\delta_i + \delta_j$
$-\epsilon_i - \epsilon_j\ (i \neq j)$	$-\epsilon_i$	$\epsilon_i - \epsilon_j$	ϵ_i	$\epsilon_i + \epsilon_j\ (i \neq j)$
$-\epsilon_i - \delta_j$		$\pm(\epsilon_i - \delta_j)$		$\epsilon_i + \delta_j$

So $\mathfrak{g}_0 = \mathfrak{gl}(m|n)$, and this gradation corresponds to the t-(super)dimension introduced earlier.

Let us consider the irreducible highest weight representation with highest weight given by $(\frac{p}{2}, \ldots, \frac{p}{2}; -\frac{p}{2}, \ldots, -\frac{p}{2})$ in the ϵ-δ-basis. This representation has Dynkin labels $[0, 0, \ldots, 0, p]$. Using $x_i = \mathrm{e}^{-\epsilon_i}$, $y_i = \mathrm{e}^{-\delta_i}$, the following character formula holds [5, 17]:

$$\mathrm{char}[0, \ldots, 0, p]_{\mathfrak{osp}(2m+1|2n)} = (y_1 \cdots y_n / x_1 \cdots x_m)^{p/2} \sum_{\lambda,\ \lambda_1 \leq p} s_\lambda(x|y). \tag{10}$$

So here the sum is over all partitions λ inside the (m, n)-hook (otherwise $s_\lambda(x|y)$ is zero anyway) with $\lambda_1 \leq p$, or equivalently $\ell(\lambda') \leq p$.

In order to determine $\mathrm{sdim}_t[0, \ldots, 0, p]_{\mathfrak{osp}(2m+1|2n)}$, one should (apart from the factor in front of the above sum) specify $x_i = t$ and $y_j = -t$ in the above character, and so one finds

$$\begin{aligned} \mathrm{sdim}_t[0, \ldots, 0, p]_{\mathfrak{osp}(2m+1|2n)} &= \sum_{\lambda,\ \lambda_1 \leq p} s_\lambda(t, \ldots, t| -t, \ldots, -t) \\ &= \sum_{\lambda,\ \lambda_1 \leq p} s_\lambda(1, \ldots, 1| -1, \ldots, -1)\, t^{|\lambda|} \\ &= \sum_{\lambda,\ \lambda_1 \leq p} \mathrm{sdim}\, V^\lambda_{\mathfrak{gl}(m|n)}\, t^{|\lambda|}. \end{aligned} \tag{11}$$

Using the properties of $\mathfrak{gl}(m|n)$ superdimensions, this leads to the following three cases.

Case 1: $m = n$, $\mathfrak{osp}(2n+1|2n)$**.** All superdimensions of covariant representations of $\mathfrak{gl}(n|n)$ are zero, except when $\lambda = (0)$. Hence:

$$\mathrm{sdim}_t[0, \ldots, 0, p]_{\mathfrak{osp}(2n+1|2n)} = 1. \tag{12}$$

Case 2: $m = n + k$, $\mathfrak{osp}(2n + 2k + 1|2n)$**.** Now it follows directly from (11) and (2) that

$$\mathrm{sdim}_t[0, \ldots, 0, p]_{\mathfrak{osp}(2m+1|2n)} = \sum_{\lambda,\ \lambda_1 \leq p} \dim V^\lambda_{\mathfrak{gl}(k)}\, t^{|\lambda|} = \sum_{\lambda,\ \lambda_1 \leq p,\ \ell(\lambda) \leq k} \dim V^\lambda_{\mathfrak{gl}(k)}\, t^{|\lambda|}. \tag{13}$$

This coincides with expression (4). Hence we can write

$$\mathrm{sdim}_t[0, 0, \ldots, 0, p]_{\mathfrak{osp}(2n+2k+1|2n)} = \dim_t[0, \ldots, 0, p]_{\mathfrak{so}(2k+1)}. \tag{14}$$

Case 3: $n = m + k$, $\mathfrak{osp}(2m + 1|2m + 2k)$**.** One finds:

$$\mathrm{sdim}_t[0, \ldots, 0, p]_{\mathfrak{osp}(2m+1|2n)} = \sum_{\lambda,\ \lambda_1 \leq p,\ \lambda_1 \leq k} (-1)^{|\lambda|} \dim V^{\lambda'}_{\mathfrak{gl}(k)}\, t^{|\lambda|} = \sum_{\mu,\ \ell(\mu) \leq \min(p,k)} \dim V^\mu_{\mathfrak{gl}(k)}\, (-t)^{|\mu|}. \tag{15}$$

The right hand side is the same expression as (8), so

$$\mathrm{sdim}_t[0,0,\ldots,0,p]_{\mathfrak{osp}(2m+1|2m+2k)} = \dim_{-t}[0,\ldots,0,-p]_{\mathfrak{osp}(1|2k)}. \tag{16}$$

So in all three cases, the superdimension for $\mathfrak{osp}(2m+1|2n)$ simplifies and reduces to a dimension of $\mathfrak{so}(2m+1-2n)$ or $\mathfrak{osp}(1|2n-2m)$.

IV. SUPERDIMENSIONS FOR $\mathfrak{osp}(2m|2n)$ AND NEW CHARACTERS

For $D(m,n) = \mathfrak{osp}(2m|2n)$, the distinguished set of simple roots in the ϵ-δ-basis is

$$\delta_1-\delta_2,\ \ldots,\delta_{n-1}-\delta_n,\ \delta_n-\epsilon_1,\ \epsilon_1-\epsilon_2,\ \ldots,\epsilon_{m-2}-\epsilon_{m-1},\ \epsilon_{m-1}-\epsilon_m,\ \epsilon_{m-1}+\epsilon_m. \tag{17}$$

It will be helpful to see $D(m,n)$ as a subalgebra of $B(m,n)$. In fact, using the $\mathbb{Z}$-gradation $\mathfrak{g}_{-2}\oplus\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_{+1}\oplus\mathfrak{g}_{+2}$ of $\mathfrak{g}=\mathfrak{osp}(2m+1|2n)$ introduced in the previous section, it is easy to see that $\mathfrak{osp}(2m|2n) = \mathfrak{g}_{-2}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_{+2}$, with root structure as in Section III.

For the irreducible highest weight representation of $\mathfrak{osp}(2m|2n)$ with highest weight given by $(\frac{p}{2},\ldots,\frac{p}{2};-\frac{p}{2},\ldots,-\frac{p}{2})$, with Dynkin labels are $[0,0,\ldots,0,p]$, the character was determined in [5]:

$$\mathrm{char}[0,\ldots,0,p]_{\mathfrak{osp}(2m|2n)} = (y_1\cdots y_n/x_1\cdots x_m)^{p/2}\sum_{\lambda\in\mathcal{B},\ \lambda_1\le p} s_\lambda(x|y). \tag{18}$$

Herein, $\mathcal{B}$ denotes the set of partitions for which each part appears twice (including the zero partition). Thus, one finds

$$\mathrm{sdim}_t[0,\ldots,0,p]_{\mathfrak{osp}(2m|2n)} = \sum_{\lambda\in\mathcal{B},\ \lambda_1\le p} \mathrm{sdim}\, V^\lambda_{\mathfrak{gl}(m|n)}\, t^{|\lambda|}. \tag{19}$$

This expression allows once again to deduce superdimension formulas in three cases: $m=n$, $m>n$ and $m<n$, see [5]. Let us give here the formula for $m>n$, i.e. $m=n+k$, or $\mathfrak{osp}(2n+2k|2n)$. From (19) one has:

$$\mathrm{sdim}_t[0,\ldots,0,p]_{\mathfrak{osp}(2m|2n)} = \sum_{\lambda\in\mathcal{B},\ \lambda_1\le p} \dim V^\lambda_{\mathfrak{gl}(k)}\, t^{|\lambda|} = \sum_{\lambda\in\mathcal{B},\ \lambda_1\le p,\ \ell(\lambda)\le k} \dim V^\lambda_{\mathfrak{gl}(k)}\, t^{|\lambda|}. \tag{20}$$

And thus, using known characters of $\mathfrak{so}(2k)$ [5]:

$$\mathrm{sdim}_t[0,0,\ldots,0,p]_{\mathfrak{osp}(2n+2k|2n)} = \begin{cases} \dim_t[0,\ldots,0,0,p]_{\mathfrak{so}(2k)} & \text{for } k \text{ even},\\ \dim_t[0,\ldots,0,p,0]_{\mathfrak{so}(2k)} & \text{for } k \text{ odd}.\end{cases} \tag{21}$$

Here, the convention for the order of the simple roots of $\mathfrak{so}(2k)$ is $\epsilon_1-\epsilon_2,\ldots,\epsilon_{k-1}-\epsilon_k,\epsilon_{k-1}+\epsilon_k$.

At this point, we can make some interesting observations and additions to the results obtained in [5]. For this, let us first consider the representations appearing here for $\mathfrak{so}(2k+1)$ and $\mathfrak{so}(2k)$. In (3) we obtained

$$\mathrm{char}[0,\ldots,0,p]_{\mathfrak{so}(2k+1)} = (x_1\cdots x_k)^{-p/2} \sum_{\lambda_1\leq p,\ \ell(\lambda)\leq k} s_\lambda(x). \tag{22}$$

Essentially, this is the branching $\mathfrak{so}(2k+1)\supset\mathfrak{gl}(k)$. But for this inclusion, there is an intermediate subalgebra: $\mathfrak{so}(2k+1)\supset\mathfrak{so}(2k)\supset\mathfrak{gl}(k)$. From Weyl's character formula, it is easy to deduce the branching of the above $\mathfrak{so}(2k+1)$ representation with respect to $\mathfrak{so}(2k)$:

$$\mathrm{char}[0,\ldots,0,p]_{\mathfrak{so}(2k+1)} = \sum_{r=0}^{p} \mathrm{char}[0,\ldots,r,p-r]_{\mathfrak{so}(2k)}. \tag{23}$$

The $\mathfrak{so}(2k)$ representations that appeared earlier, with expressions in terms of Schur functions, were $[0,\ldots,0,p]$ and $[0,\ldots,0,p,0]$. So the question is now: how to write the character of the other $\mathfrak{so}(2k)$ representations $[0,\ldots,r,p-r]$ as a sum of Schur functions? Or in other words, what is the branching $\mathfrak{so}(2k)\supset\mathfrak{gl}(k)$ for these representations? The answer is:

Theorem. *For k even, one has*

$$\mathrm{char}[0,\ldots,0,r,p-r]_{\mathfrak{so}(2k)} = (x_1\cdots x_k)^{-p/2} \sum_{\lambda_1\leq p,\ \ell(\lambda)\leq k;\ \lambda\in\mathcal{B}_r} s_\lambda(x). \tag{24}$$

Herein, $\mathcal{B}_r$ stands for the set of partitions of $\mathcal{B}$ to which a horizontal strip of length r is attached. (Recall that $\mathcal{B}$ is the set of partitions for which each part appears twice.) The first condition ($\lambda_1\leq p$, $\ell(\lambda)\leq k$) means that (the Young diagram of) λ fits inside the $k\times p$ rectangle. Similarly, for k odd:

$$\mathrm{char}[0,\ldots,0,r,p-r]_{\mathfrak{so}(2k)} = (x_1\cdots x_k)^{-p/2} \sum_{\lambda_1\leq p,\ \ell(\lambda)\leq k;\ \lambda\in\mathcal{B}_{p-r}} s_\lambda(x). \tag{25}$$

We have not found the above result in the literature. The actual proof is rather technical. It can be obtained using the branching rules for $\mathfrak{so}(2k)\supset\mathfrak{gl}(k)$ described in [18]. Note that, in accordance with (23), the union of all partitions of $\mathcal{B}_r$ in the $k\times p$ rectangle, for $r=0,1,\ldots,p$, is equal to the set of all partitions in the rectangle.

But now we can extend the analogy that we observed between representations $[0,\ldots,0,p]$ of $\mathfrak{osp}(2m|2n)$ and those of $\mathfrak{so}(2k)$ for $m=n+k$. This leads to the following

Conjecture. *For $|m-n|$ even, one has*

$$\mathrm{char}[0,\ldots,0,r,p-r]_{\mathfrak{osp}(2m|2n)} = (y_1\cdots y_n/x_1\cdots x_m)^{p/2} \sum_{\lambda_1\leq p,\ \lambda\in\mathcal{B}_r} s_\lambda(x/y). \tag{26}$$

So in this case we have an expansion as an infinite sum of supersymmetric Schur functions, labeled by partitions λ inside the (m,n)-hook, of width at most p, and belonging to $\mathcal{B}_r$.

For $|m-n|$ odd, the result is similar, with $\mathcal{B}_r$ replaced by $\mathcal{B}_{p-r}$.

To conclude the paper, we have analyzed characters and superdimensions for representations of the form $[0,\ldots,0,p]$ for $\mathfrak{osp}(2m+1|2n)$, and of the form $[0,\ldots,0,r,p-r]$ for $\mathfrak{osp}(2m|2n)$. It should be noted that characters for more general $\mathfrak{osp}(m|n)$ tensors have been studied in [19]. However, the formulas in [19] lead to alternating series of S-functions, which are not as easy to handle as the characters obtained here.

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